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THE CLASSIFICATION OF 4-DIMENSIONAL P-ADIC FILIFORM LEIBNIZ ALGEBRAS

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Abstract. In this paper Leibniz algebras over the field of p-adic numbers are investigated and 4-dimensional p-adic filiform Leibniz algebras are classified.

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1. Introduction

The notion of Leibniz algebras was introduced in 90-th by J-L. Loday [6] as a "non antisymmetric" generalization of Lie algebras. Later the structure theory of these algebras has been investigated by various authors [1, 2, 3]. In these papers the Leibniz algebras were considered over the field of complex numbers. During last decades various mathematical structures have been considered over the field of p-adic numbers: p-adic functional analysis, p-adic differential equations, p-adic probability theory, p-adic mathematical physics etc. The present paper is devoted to study of finite dimensional Leibniz algebras over the field of p-adic numbers.

The classification of low dimensional algebras is an important step in investigation of finite dimensional algebras. Recall that the description of finite dimensional complex Lie algebras has been reduced to the classification of nilpotent Lie algebras, which have been completely classified up to dimension 7. The problem of classification of complex Leibniz algebras has been solved for dimensions up to 3. The description of 3-dimentional solvable p-adic Leibniz algebras [5] shows that even in this case the list of 3-dimentional p-adic Leibniz algebras is essentially wider than in the complex case.

In the present paper we describe p-adic filiform Leibniz algebras of dimension 4. Recall that in the description of complex filiform Leibniz algebras it was sufficient to consider special type basis transformations [2]. We modify some results of the complex case in order to describe p-adic filiform Leibniz algebras.

2. Preliminaries and notations

Let $\mathbb{Q}$ be the field of rational numbers and let p be an arbitary but fixed prime number. Each rational number $x \neq 0$ is represented in the form $x = p^{\gamma(x)}\dfrac{n}{m}$, where $m \in \mathbb{N}$, and $n, \gamma(x) \in \mathbb{Z}$, the integers m, n do not have p as a factor, i.e. $p \nmid m$ and $p \nmid n$.

Equip the field $\mathbb{Q}$ by the following p-adic norm

$$|0|_p = 0, \ \ |x|_p = p^{-\gamma(x)}, \ x \neq 0.$$

The norm $|\ |_p$ satisfies the ultrametric inequality $|x+y|_p \leq max(|x|_p, |y|_p)$, $x, y \in \mathbb{Q}$.

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The completion of the field $\mathbb{Q}$ in this non archimedean norm forms a field which is called *the field of p-adic numbers* and denoted by $\mathbb{Q}_p$ (see for details [4, 9]).

It is known that each p-adic number $x \neq 0$ can be uniquely expanded in the following canonical form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...)$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, x_j- integers such that $x_0 > 0$, $0 \leq x_j \leq p-1$, $(j = 1, 2, \dots)$.

p-adic numbers x which satisfy $|x|_p \leq 1$ are called *p-adic integers*. The set of all p-adic integers is denote by $\mathbb{Z}_p$. p-adic integers $x \in \mathbb{Z}_p$ with $|x|_p = 1$ are called *p-adic units* in $\mathbb{Z}_p$.

Recall that an integer $x \in \mathbb{Z}$ is said to be *quadratic residue modulo p* if the equation

$$x^2 \equiv a(mod\ p)$$

has a solution $x \in \mathbb{Z}$, otherwise a is called *a quadratic non residue modulo p*.

Now consider a p-adic number $a \in \mathbb{Q}_p$ $(a \neq 0)$ with the canonical expansion

$$a = p^{\gamma(a)}(a_0 + a_1 p + ...), a_0 > 0, 0 \leq a_j \leq p-1, j = 1, 2, ...$$

Lemma 2.1. [9] *The equation*

$$x^2 = a$$

has a solution $x \in \mathbb{Q}_p$ if and only if the following two conditions are satisfied:

1) $\gamma(a)$ is an even integer,

2) a_0 is a quadratic residue modulo p, if $p \neq 2$; $a_1 = a_2 = 0$, if $p = 2$.

Let η be a p-adic number which is not the square of any p-adic number (i.e. η has no square root in $\mathbb{Q}_p$). Then Lemma 2.1 implies

Corollary 2.1.[9] *For $p \neq 2$, the numbers $\varepsilon_1 = \eta$, $\varepsilon_2 = p$, $\varepsilon_3 = p\eta$ have no square roots in $\mathbb{Q}_p$. Every p-adic number x can be represented in one of the following four forms:*

$$x = \varepsilon_j y_j^2,\ 0 \leq j \leq 3,$$

where $y_j \in Q_p$ and $\varepsilon_0 = 1$, $\varepsilon_1 = \eta$, $\varepsilon_2 = p$, $\varepsilon_3 = p\eta$ have no square roots in $\mathbb{Q}_p$.

Corollary 2.2.[9] *For $p = 2$, the numbers $\varepsilon_j \in \{2, 3, 5, 6, 7, 10, 14\}$ $(1 \leq j \leq 7)$ and their mutual products have no square roots in the field of 2-adic numbers. Each 2-adic number can be represented in one of the following eight forms $x = \varepsilon_j y_j^2$, where $\varepsilon_0 = 1$, $y_j \in Q_p$, $1 \leq j \leq 7$.*

Now let us turn to Leibniz algebras.

Definition 2.1. *An algebra L over a field F is said to be a* Leibniz algebra *if the multiplication $[\ ,\ ]$ in L satisfies the following Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

for all $x, y, z \in L$.

It should be noted that if a Leibniz algebra L satisfies the antisymmetricity condition $[x, x] = 0$ for every $x \in L$, then the Leibniz identity coincides with the Jacobi identity. Therefore Leibniz algebras are non antisymmetric generalizations of Lie algebras.

Given a Leibniz algebra consider the following sequence:

$$L^1 = L, L^{k+1} = [L^k, L],\ \ k \geq 1.$$

Definition 2.2. *An n-dimensional Leibniz algebra L is said to be* filiform *if $dimL^i = n - i$, $2 \leq i \leq n$.*

It should be noted that this definition agrees with the definition of filiform Lie algebras.

Now define the natural gradation for a filiform Leibniz algebra L.

Put $L_i = L^i/L^{i+1}$, then $dimL_1 = 2$ and $dimL_i = 1$, $2 \leq\ i \leq n-1$. Consider the space $grL := L_1 \oplus L_2 \oplus ... \oplus L_{n-1}$. In view of the inclusions $[L^i, L^j] \subset L^{i+j}$ it is easy to see that $[L_i, L_j] \subset L_{i+j}$, i.e. one has a gradation.

Definition 2.3. *A filiform Leibniz algebra L is said to be* naturally graded *if $L \cong grL$.*

From now on we shall consider Leibniz algebra over the field $\mathbb{Q}_p$ of p-adic numbers, i.e. *p-adic Leibniz algebras.*

A careful analysis of the results in [2, 3] and [5] shows that similar to the complex case, the p-adic filiform Leibniz algebras are divided into three disjoint classes of algebras.

Theorem 2.1. *In an arbitrary n-dimensional p-adic filiform Leibniz algebra there exists a basis $\{e_1, e_2, ..., e_n\}$ such that the multiplication in this basis has one of the following three forms:*

$$
\begin{array}{lll}
a) F_n^1: & [e_1,e_1]=e_3, [e_i,e_1]=e_{i+1}, & 2 \le i \le n-1, \\
& [e_1,e_2]=\alpha_4 e_4+\alpha_5 e_5+...+\alpha_{n-1}e_{n-1}+\theta e_n, & \\
& [e_j,e_2]=\alpha_4 e_{j+2}+\alpha_5 e_{j+3}+...+\alpha_{n+2-j}e_n, & 2 \le j \le n-2,
\end{array}
$$

where $\alpha_i, \theta \in \mathbb{Q}_p$ and the omitted of products are equal to zero;

$$
\begin{array}{lll}
b) F_n^2: & [e_1,e_1]=e_3, [e_i,e_1]=e_{i+1}, & 3 \le i \le n-1, \\
& [e_1,e_2]=\beta_4 e_4+\beta_5 e_5+...+\beta_n e_n, [e_2,e_2]=\gamma e_n & \\
& [e_j,e_2]=\beta_4 e_{j+2}+\beta_5 e_{j+3}+...+\beta_{n+2-j}e_n, & 3 \le j \le n-2,
\end{array}
$$

where $\beta_i, \gamma \in \mathbb{Q}_p$ and the omitted products being zero;

$$
\begin{array}{lll}
c) F_n^3: & [e_i,e_1]=e_{i+1}, & 2 \le i \le n-1, \\
& [e_1,e_{i+1}]=-e_{i+1} & 3 \le i \le n-1, \\
& [e_1,e_1]=\theta_1 e_n, [e_1,e_2]=-e_3+\theta_2 e_n, [e_2,e_2]=\theta_3, & \\
& [e_2,e_j]=-[e_j,e_2] \in \{e_{j+2}, e_{j+3}, ..., e_n\} & 3 \le j \le n-2, \\
& [e_i,e_j]=-[e_j,e_i] \in \{e_{i+j}, e_{i+j+1}, ..., e_n\} & 3 \le i \le [\frac{n}{2}], i \le j \le n-i,
\end{array}
$$

where $\theta_i \in \mathbb{Q}_p$ and the above products satisfy the Leibniz identity.

Definition 2.4. *Define the following types of basis transformation:*

$$
\vartheta(a,b) = \begin{cases}
f(e_1) = ae_1 + be_2, & \\
f(e_2) = (a+b)e_2 + b(\theta - \alpha_n)e_{n-1}, & \\
f(e_{i+1}) = [f(e_i), f(e_1)], & 2 \le i \le n-1, \\
f(e_3) = [f(e_1), f(e_1)], & a(a+b) \neq 0,
\end{cases}
$$

$$
\delta(a,b,d) = \begin{cases}
f(e_1) = a_1 e_1 + b_1 e_2, & \\
f(e_2) = de_2 - \dfrac{bd\gamma}{a} e_{n-1}, & \\
f(e_{i+1}) = [f(e_i), f(e_1)], & 3 \le i \le n-1, \\
f(e_3) = [f(e_1), f(e_1)], & ad \neq 0,
\end{cases}
$$

where $a, b, d \in \mathbb{Q}_p$.

The investigation of filiform Leibniz algebras over the field of p-adic numbers similarly to the complex case can be reduced to the study of the transformations ϑ and δ respectively for the first and the second classes of algebras from Theorem 2.1.

3. Classification of 4-dimensional p-adic filiform Leibniz algebras

Let η be a p-adic unit, which has no square root in $\mathbb{Q}_p$.

Lemma 3.1. *For any $p \ge 3$ the square root $\sqrt{4+p^2\varepsilon}$ exists in $\mathbb{Q}_p$, where $\varepsilon \in \{1, \eta, p, p\eta\}$. The square root $\sqrt{4+8^2\varepsilon}$ exists in $\mathbb{Q}_2$, where $\varepsilon \in \{1, 2, 3, 5, 6, 7, 10, 14\}$.*

Proof. Let $p \geq 3$. We have to show that $\sqrt{4+p^2\varepsilon}$ exists in $\mathbb{Q}_p$, where $\varepsilon \in \{1, \eta, p, p\eta\}$. Consider the canonical expansion of the number ε:

$$\varepsilon = p^{\gamma(\varepsilon)}(\varepsilon_0 + \varepsilon_1 p + \varepsilon_2 p^2 + ...), \tag{1}$$

where $\gamma(\varepsilon) \in \mathbb{Z}$, $\varepsilon_0 > 0$, $\varepsilon \in \{0, 1, 2, ..., p-1\}$, $i = 0, 1, ...$

Then

$$4 + p^2\varepsilon = 4 + p^{2+\gamma(\varepsilon)}(\varepsilon_0 + \varepsilon_1 p + \varepsilon_2 p^2 + ...). \tag{2}$$

If $p \neq 2$, $\varepsilon \in \{1, \eta, p, \eta p\}$, the numbers 1 and η are p-adic units, i.e. $|1|_p = |\eta|_p = 1$. Therefore the numbers p and $p\eta$ are integers and $|p|_p = |\eta p|_p = \dfrac{1}{p}$. Since p and ηp are p-adic integers, $\gamma(\varepsilon)$ is equal either to 0 or to 1.

Let $p = 3$. Then rewrite the expansion (2) in the form:

$$4 + 3^2\varepsilon = 1 + 3 + 3^{2+\gamma(\varepsilon)}(\varepsilon_0 + \varepsilon_1 3 + \varepsilon_2 3^2 + ...).$$

By Lemma 2.1 the equation $x^2 \equiv 1(mod\ 3)$ is solvable in $\mathbb{Z}$, namely $x = 3N + 1$, $N \in \mathbb{Z}$. Therefore $\sqrt{4+p^2\varepsilon}$ exists in $\mathbb{Q}_p$.

Let $p \geq 5$, then

$$4 + p^2\varepsilon = 4 + p^{2+\gamma(\varepsilon)}(\varepsilon_0 + \varepsilon_1 p + \varepsilon_2 p^2 + ...).$$

Lemma 2.1 implies that the equation $x^2 \equiv 4(mod\ p)$ has a solution in $\mathbb{Z}$, namely $x = pN + 2$, $N \in \mathbb{Z}$. Therefore $\sqrt{4+p^2\varepsilon}$ exists in $\mathbb{Q}_p$.

Let $p = 2$ and let us show that $\sqrt{4+8^2\varepsilon}$ exists in $\mathbb{Q}_p$, where $\varepsilon \in \{1, 2, 3, 5, 6, 7, 10, 14\}$. Since 1, 3, 5, 7 are p-adic units and 2, 6, 10, 14 are p-adic integers ($|2|_p = |6|_p = |10|_p = |14|_p = \dfrac{1}{2} < 1$), we have that in the expansion

$$\varepsilon = p^{\gamma(\varepsilon)}(\varepsilon_0 + \varepsilon_1 p + \varepsilon_2 p^2 + ...),$$

the integer $\gamma(\varepsilon)$ is equal either to 0 or to 1. Therefore the existence of $\sqrt{4+8^2\varepsilon}$ follows from Lemma 2.1.

Now recall the notion of decomposable Leibniz algebra. An algebra L is said to be *decomposable*, if there exist two subalgebras N and M in L such that $L = M \oplus N$ and $[M, N] = [N, M] = \{0\}$. Therefore the classification of decomposable Leibniz algebras can be reduced to lower dimensional cases. So from now on we shall consider only non decomposable Leibniz algebras.

The following theorem gives a classification of non decomposable 4-dimensional p-adic filiform Leibniz algebras.

Theorem 3.1. *An arbitrary 4-dimensional filiform non decomposable Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$L_1:$ $[e_1, e_1] = e_3,$ $[e_1, e_2] = e_4,$ $[e_2, e_1] = e_3,$ $[e_3, e_1] = e_4,$
$L_2:$ $[e_1, e_1] = e_3,$ $[e_2, e_1] = e_3,$ $[e_3, e_1] = e_4,$
$L_3:$ $[e_1, e_1] = e_3,$ $[e_1, e_2] = e_4,$ $[e_3, e_1] = e_4,$
$L_4:$ $[e_1, e_1] = e_3,$ $[e_2, e_1] = e_3,$ $[e_1, e_2] = e_4,$ $[e_3, e_1] = e_4,$ $[e_2, e_2] = e_4,$
$L_5:$ $[e_1, e_1] = e_3,$ $[e_2, e_1] = e_3,$ $[e_3, e_1] = e_4,$ $[e_2, e_2] = e_4,$
$L_6(\varepsilon):$ $[e_1, e_1] = e_3,$ $[e_3, e_1] = e_4,$ $[e_2, e_2] = \varepsilon e_4,$

$L_7:$ $[e_2,e_1]=e_3,$ $[e_1,e_2]=-e_3,$ $[e_3,e_1]=e_4,$ $[e_1,e_3]=-e_4,$
$L_8:$ $[e_1,e_1]=e_4,$ $[e_2,e_1]=e_3,$ $[e_3,e_1]=e_4,$ $[e_1,e_2]=-e_3,$
$[e_1,e_3]=-e_4,$
$L_9:$ $[e_1,e_1]=e_4,$ $[e_2,e_1]=e_3,$ $[e_3,e_1]=e_4,$ $[e_1,e_2]=-e_3+e_4,$
$[e_1,e_3]=-e_4,$
$L_{10}:$ $[e_1,e_1]=e_4,$ $[e_2,e_2]=e_4,$ $[e_2,e_1]=e_3,$ $[e_1,e_2]=-e_3+2e_4,$
$[e_3,e_1]=e_4,$ $[e_1,e_3]=-e_4,$
$L_{11}(\varepsilon):$ $[e_1,e_1]=e_4,$ $[e_2,e_2]=e_4,$ $[e_2,e_1]=e_3,$ $[e_1,e_2]=-e_3+\sqrt{4+\alpha^2\varepsilon}e_4,$
$[e_3,e_1]=e_4,$ $[e_1,e_3]=-e_4,$

where $\alpha=p$ *and* $\varepsilon\in\{1,\eta,p,p\eta\}$ *if* $p\neq 2$; $\alpha=8$ *and* $\varepsilon\in\{1,2,3,5,6,7,10,14\}$ *if* $p=2$.

Proof. Let L be a 4-dimensional p-adic filiform Leibniz algebra. By Theorem 2.1 there exists a basis $\{e_1,e_2,e_3,e_4\}$ in which the algebra has one of the following three forms:

$F_n^1:$ $[e_1,e_1]=e_3,$ $[e_1,e_2]=e_4,$ $[e_2,e_1]=e_3,$ $[e_2,e_2]=e_4,$ $[e_3,e_1]=e_4.$

$F_n^2:$ $[e_1,e_1]=e_3,$ $[e_1,e_2]=e_4,$ $[e_2,e_2]=e_4,$ $[e_3,e_1]=e_4.$

$F_n^3:$ $[e_1,e_1]=e_4,$ $[e_2,e_2]=e_4,$ $[e_2,e_1]=e_3,$ $[e_3,e_1]=e_4,$ $[e_1,e_2]=-e_3+e_4,$
$[e_1,e_3]=-e_4.$

Consider the algebra F_4^1. Based on properties of the transformations ϑ and δ from Theorem 2.1 consider the following change of the basis:

$e_1'=ae_1+be_2,$
$e_2'=(a+b)e_2+b(\alpha-\beta)e_3,$
$e_3'=[e_2',e_1']=[(a+b)e_2+b(\alpha-\beta)e_3,ae_1+be_2]=a(a+b)e_3+(a(\alpha-\beta)+b\beta(a+b))e_4,$
$e_4'=[e_3',e_1']=[a(a+b)e_3+(a(\alpha-\beta)+b\beta(a+b))e_4,ae_1+be_2]=a^2(a+b)e_4,$
where $a^2(a+b)\neq 0.$

Writing down the multiplication in the algebra from the class F_4^1 in the new basis, and comparing the coefficients of the basic elements $\{e_1,e_2,e_3,e_4\}$ we obtain the following relations:

$$\alpha'a^2=\alpha a+\beta b \text{ and } \beta'a^2=\beta(a+b).$$

Case 1 Let $\beta=0$, then $\beta'=0$ and $\alpha'a=\alpha.$

If $\alpha\neq 0$, then putting $a=\alpha$, we obtain $\alpha'=1$ and come to the algebra L_1.

Otherwise, i.e. if $\alpha=0$, we have $\alpha'=0$ and obtain the algebra L_2.

Case 2. Let $\beta\neq 0$. Then putting $b=\dfrac{a^2}{\beta}-a$, we obtain $\beta'=1$ and $\alpha'=\dfrac{a^2+a\alpha-a\beta}{a^2}$.

If $\alpha=\beta$, then $\alpha'=1$ and we obtain the algebra L_3.

If $\alpha\neq\beta$, then putting $a=\beta-\alpha$ we have $\alpha'=0$ and come to the algebra L_4.

Now let us consider the class of algebras F_4^2. By considering the transformations ϑ and δ for the classes a) and b) from Theorem 2.1 take the change of basis in the form:

$e_1'=ae_1+be_2,$
$e_2'=ce_2-ba^{-1}e_3,$
$e_3'=[e_2',e_1']=(ae_1+be_2)(ce_2-ba^{-1}e_3)=a^2e_3+b(\alpha a+\beta b)e_4,$
$e_4'=[e_3',e_1']=(a^2e_3+b(\alpha a+\beta b))e_4)(ae_1+be_2)=a_3e_4,$ where $a^3\neq 0.$

Writing down the multiplication in the algebra from the class F_4^2 in the new basis and comparing the coefficients of the basic elements $\{e_1,e_2,e_3,e_4\}$ we obtain the following relations:

$$\alpha'a^3=c(\alpha a+\beta b) \text{ and } \beta'a^3=\beta c^2.$$

Case 1. Let $\beta=0$. Then $\beta'=0$ and $\alpha'a^2=c\alpha.$

If $\alpha=0$, then we have $\alpha'=0$ and obtain the algebra with the product

$$[e_1,e_1]=e_3,[e_3,e_1]=e_4.$$

It is easy to see that this algebra is decomposable.

If $\alpha \neq 0$, then putting $c = \dfrac{a^2}{\alpha}$, we have $\alpha' = 1$ and we obtain the algebra L_5.

Case 2. Let $\beta \neq 0$. By Corollaries 2.1 and 2.2 the parameter $\dfrac{\beta}{a^3}$ can be represented in the form $\dfrac{\beta}{a^3} = \varepsilon y_j^2$. If we take $c = \dfrac{1}{y_j}$, then $\beta' = \varepsilon$, where $\varepsilon \in \{1, \eta, p, p\eta\}$ if $p \neq 2$; and $\varepsilon \in \{1, 2, 3, 5, 6, 7, 10, 14\}$ if $p = 2$. Thus we obtain the algebras $L_6(\varepsilon)$, which are mutually non isomorphic for different ε.

Now consider the third class of algebras F_4^3.

If $(\alpha, \beta, \gamma) = (0, 0, 0)$, then this algebra is a Lie algebra and we have the case L_7.

Suppose that $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. If $\alpha = 0$, then $(\beta, \gamma) \neq (0, 0)$ and by changing the basis as $e_1' = e_1 + Ae_2$ we have $[e_1, e_1] = A(\beta + A\gamma)e_4 \neq 0$, i.e. given any β, γ there exists $A \neq 0$ such that $\beta + A\gamma \neq 0$. Therefore without loss of generality we may assume that $\alpha \neq 0$.

So let $\alpha \neq 0$. Then by changing the basis as

$$e_1' = \alpha e_1, e_2' = \alpha e_1, e_3' = \alpha^2 e_3, e_4' = \alpha^3 e_4,$$

we obtain $\alpha = 1$ and the multiplication has the form

$$[e_1, e_1] = e_4, [e_2, e_1] = e_3, [e_2, e_2] = \alpha e_4, [e_1, e_2] = -e_3 + \beta e_4,$$
$$[e_3, e_1] = -[e_1, e_3] = e_4. \tag{3}$$

Let us consider the problem of mutual isomorphism of algebras from this family. Take a general change of basis:

$e_1' = A_1e_1 + A_2e_2 + A_3e_3$,

$e_2' = B_1e_1 + B_2e_2 + B_3e_3$,

$e_3' = [e_2', e_1'] = [B_1e_1 + B_2e_2 + B_3e_3, A_1e_1 + A_2e_2 + A_3e_3] = (A_1B_2 - A_2B_1)e_3 + (A_1B_1 + A_1B_3 + \beta A_2B_1 + \alpha A_2B_2 - A_3B_1)e_4$,

$e_4' = [e_3', e_1'] = [(A_1B_2 - A_2B_1)e_3 + (A_1B_1 + A_1B_3 + \beta A_2B_1 + \alpha A_2B_2 - A_3B_1)e_4, A_1e_1 + A_2e_2 + A_3e_3] = A_1(A_1B_2 - A_2B_1)e_4$, where $A_1(A_1B_2 - B_1A_1) \neq 0$.

Writing down the multiplication in the algebra from the family (3) in the new basis and comparing the coefficients of the basic elements $\{e_1, e_2, e_3, e_4\}$, we obtain the following relations:

$$A_1^2 + \beta A_1A_2 + \alpha A_2^2 = A_1^2B_2, \ \ B_1 = 0, \ \ A_1^2B_2 \neq 0,$$

$$\alpha' = \frac{B_2\alpha}{A_1^2}, \ \ \beta' = \frac{A_1\beta + 2A_2\alpha}{A_1^2}.$$

It is easy to calculate that

$$\beta'^2 - 4\alpha' = \frac{1}{A_1^2}(\beta^2 - 4\alpha).$$

Case 1. Let $\alpha = 0$, then $\alpha' = 0$.

Subcase 1.1. If $\beta = 0$, then $\beta' = 0$ and we obtain the algebra L_8.

Subcase 1.2. If $\beta \neq 0$, then putting $A_1 = \beta$, $B_2 = 1$, $A_2 = 0$ we have $\beta' = 1$ and we come to the algebra L_9.

Case 2. Let $\alpha \neq 0$. Then putting $B_2 = \dfrac{A_1^2}{\alpha}$, we obtain $\alpha' = 1$.

Subcase 2.1. If $\beta^2 - 4\alpha = 0$, then putting $A_2 = \dfrac{1}{2\alpha}(2A_1^2 - A_1\beta)$ and $A_1 \neq 0$, we have $\beta' = 2$ and obtain the algebra L_{10}.

Subcase 2.2. If $\beta^2 - 4\alpha \neq 0$, then by Corollaries 2.1 and 2.2 it follows that $\beta^2 - 4\alpha$ can be represented as $\beta^2 - 4\alpha = \varepsilon y_j^2$, where $y_j \in \mathbb{Q}_p$. Since the square roots $\sqrt{4 + p^2\varepsilon}$ and $\sqrt{4 + 8^2\varepsilon}$ exist respectively in $\mathbb{Q}_p$ and $\mathbb{Q}_2$ by Lemma 3.1, then by putting $A_1 = \dfrac{y_j}{\alpha}$ and $A_2 =$

$\frac{1}{2\beta}(\frac{y_j^2}{\alpha^2}\sqrt{4+p^2\varepsilon}-\gamma\frac{y_j}{\alpha})$, we obtain $\gamma' = \sqrt{4+p^2\varepsilon}$, where $\varepsilon \in \{1,\eta,p,\eta p\}$ for $p \neq 2$. If $= 2$, then by putting $A_1 = \frac{y_j}{\alpha}$ and $A_2 = \frac{1}{2\beta}(\frac{y_j^2}{\alpha^2}\sqrt{4+8^2\varepsilon}-\gamma\frac{y_j}{\alpha})$, we obtain $\gamma' = \sqrt{4+8^2\varepsilon}$, where $\varepsilon \in \{1,2,3,5,6,7,10,14\}$. Therefore we have the algebra $L_{11}(\varepsilon)$. The proof of Theorem 3.1 is complete.

Remark. In the complex case one has almost a similar list of 4-dimensional filiform Leibniz algebras [1]. The main difference is that in the p-adic case we have two additional algebras $L_6(\varepsilon)$ and $L_{11}(\varepsilon)$.

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